arXiv:hep-th/9412118v1 14 Dec 1994

Modular Invariance and (Quasi)-Galois symmetry in Conformal Field Theory*

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Abstract

A brief heuristic explanation is given of recent work [1,2] with Jürgen Fuchs, Beatriz Gato-Rivera and Christoph Schweigert on the construction of modular invariant partition functions from Galois symmetry in conformal field theory. A generalization, which we call quasi-Galois symmetry [3], is also described. As an application of the latter, the invariants of the exceptional algebras at level g (for example E_8 level 30) expected from conformal embeddings are presented.

$\star$ Presented at the International Symposium on the Theory of Elementary Particles Wendisch-Rietz, August 30 - September 3, 1994

1. Introduction

Since the title of this conference mentions "elementary particle physics", I will begin by sketching the relation between that field and the subject of this talk. The usual argument goes like this: "Elementary particles and their interactions are described by string theory. String theory may be unique, but it has an extremely large number of vacua, and at most one of those vacua corresponds to the standard model. Each vacuum corresponds to a two-dimensional conformal field theory. In addition to conformal invariance, such a theory may have additional invariances, and it may happen that the symmetry is so powerful that the entire content of the theory can be organized in terms of a finite number of representations. Then the conformal field theory is called rational. Knowing all rational conformal field theories may give sufficient information about the full set of conformal field theories, either because the rational conformal field theories cover the parameter space densely, or at least because one can reach all theories by making small perturbations around the rational ones. All rational conformal field theories are cosets G/H (where G is a WZW-model and H a gauged subgroup) or orbifolds of cosets. The problem of classifying all string vacua begins thus with the classification of all ungauged WZW-models."

Many of the statements in the foregoing line of arguments are highly questionable. We don't know whether string theory, or our present way of thinking about it, has anything to do with nature, nor whether RCFT can give us sufficient information about all its vacua. The classification of RCFT's is still an unsolved problem, and the conjecture that they are all somehow related to coset theories at least requires some amendments. In addition, in many cases we do not even know how to work out the characters, the partition functions, the fusion rules and even the spectrum of coset theories because of fixed points in the field identification. Furthermore solving the classification problem for WZW models does not solve the classification problem for cosets. Finally, the classification problem of WZW itself is probably unsolvable. It may well be solved during our lifetime for the simple WZW models, but this does not imply a classification for semi-simple models. The classification problem does not factorize, and entirely new solutions can appear for tensor products of simple theories that cannot be anticipated, as far as we know.

Nevertheless, the classification problem of WZW-models has intrigued several

people. Before asking why, let us formulate the problem. Algebraically, a WZW-model is described in terms of a combination of the Virasoro algebra of conformal transformations, generated by currents of spin 2, and a Kac-Moody algebra generated by spin 1 currents. The generators T_n^a of a Kac-Moody algebra satisfy

$$[T_n^a, T_m^b] = i f^{abc} T^c + \tfrac{1}{2} k m \delta_{n,-m} \delta^{ab} \ ,$$

where f^{abc} are the structure constants of a Lie-algebra, and k an integer (in fact there such an integer for every simple factor of the Lie-algebra.) All (untwisted, simple) KM-algebras are thus characterized by a type $T = A, B, C, D, E, F$ or G, a rank r and a level k, and they will be denoted $T_{r,k}$.

There is some additional freedom in putting the conformal field theory together. At one loop level, the differences manifest themselves in the partition function. One-loop partition functions can be written on the one hand as path-integral of the conformal field theory on a torus, and on the other hand as a trace over the exponentiated Hamiltonian. The latter expression has the form

$$P(\tau, \bar{\tau}) = \text{Tr}\, e^{2\pi i \tau H_L} e^{-2\pi i \bar{\tau} H_R} \ ,$$

where H_L and H_R are the Hamiltonians (the zero-mode generators of the conformal transformations, up to a constant) of the left- and right-moving modes on the two-dimensional surface. Their eigenvalues are called conformal weights.

The parameter τ describes the inequivalent shapes of the torus. Equivalences due to reparametrizations or rescalings have been removed from the path integral by gauge-fixing. In string theory one must integrate over this parameter, but it turns out that not all values of τ are really inequivalent. There are still some global reparametrizations that have not been removed yet. This divides the complex upper half plane in which τ lives in an infinite number of equivalent regions, and if one integrates over one such region all tori have been properly included. However, this procedure is only correct if the integral does not depend on the region.

The different regions are mapped into each other by a discrete group called the modular group, which is isomorphic to $SL(2, \mathbf{Z})$. This group is generated by two

transformations

$$S : \tau \to -\frac{1}{\tau}$$
$$T : \tau \to \tau + 1$$

The requirement of modular invariance is now that $P(\tau, \bar{\tau})$ is invariant under these two transformations, and hence under the full group. [The requirement that not just the integral, but the integrand itself should be invariant arises since we should not just consider the one-loop vacuum amplitude, but all correlators at any loop order].

Using the symmetries of WZW we can combine all physical states into a finite number of representations, each of which is infinite-dimensional. Then the partition function takes the following form

$$P(\tau, \bar{\tau}) = \sum_{i,j} M_{ij} \mathcal{X}_i(\tau) \mathcal{X}_j^*(\bar{\tau}) \ ,$$

where M_{ij} is a set of non-negative integer multiplicities for the left representation i with the right representation j, and $\mathcal{X}_i$ is the (Virasoro) character of the representation i. These characters transform in the following way under modular transformations

$$S : \mathcal{X}_i \to S_{ij} \mathcal{X}_j$$
$$T : \mathcal{X}_i \to T_{ij} \mathcal{X}_j \ ,$$

where T is a diagonal unitary matrix and S a symmetric unitary matrix. The condition for modular invariance is now simply

$$[S, M] = [T, M] = 0 \ , \tag{1.1}$$

combined with the condition that M be integer and non-negative. Furthermore to have a unique ground state (denoted by "0") one requires that $M_{00} = 1$.

At higher loops some additional requirements have to be satisfied, but this is not expected to affect the result very strongly. Nevertheless one should keep in mind that solving the purely algebraic conditions (1.1) does not yet guarantee the existence of a corresponding conformal field theory.

A trivial solution to (1.1) is $M = \mathbf{1}$, which is called the diagonal invariant. Several years ago, Cappelli, Itzykson and Zuber [4] found the general solution for the Lie-algebra A_1 at arbitrary level k. It might have seemed reasonable to expect the general solution to be known within a year or so, but this has not happened. Just two years ago Gannon [5] completed the classification for A_2 and arbitrary k, but although some further progress was made, the complete classification problem is still far from solved, even for simple algebras.

Why bother? As explained earlier, it would be ridiculous to claim that solving this problem would produce a tremendous breakthrough in understanding the vacuum structure in string theory. Nor would it be of overwhelming importance for statistical mechanics, where the same problems were formulated, mainly due to the work of Cardy. The attraction of the problem is probably somewhat similar to that of Fermat's conjecture: very simple to formulate, yet very hard to solve. This is not to suggest that the problem of classifying all simple WZW modular invariants is as important, as deep, or as difficult as Fermat's problem, only that the motivation is somewhat similar. Just as in that case, the problem itself may be far less important than the new insights one gains and the methods one develops while trying to solve it.

There is a good example of that already. A very general class of non-diagonal modular invariants is the one generated by simple currents [6]. Simple currents were discovered by studying WZW modular invariants. In the special case of WZW models almost all simple currents correspond to automorphisms of the extended Dynkin diagrams, and the partition functions can be viewed in terms of strings on non-simply connected group manifolds, obtained by identifying points on the universal covering group related by elements of the center of the group [7]. But the notion of simple currents abstracted from this is more general, and already WZW models themselves provide an example, namely the simple current of $E_{8,2}$, which has no known interpretation in terms of global properties of the group manifold (among WZW-models this is the only exception [8], but one exception is sufficient to prove the point).

Galois symmetry could have been discovered in a similar way, although historically this is not what happened. Our own involvement [9] with this subject started

with an observation about the fusion rules. The fusion rules of a CFT give the number of allowed three-point couplings between certain representations. They have the form

$$R_i \times R_j = \sum_k N_{ij}^k R_k \ ,$$

where R_i is a representation (or a "primary field") and N_{ij}^k a non-negative integer.

It was pointed out by E. Verlinde [10] that N_{ij}^k can be written in terms of the matrix S:

$$N_{ij}^k = \sum_n \frac{S_{in} S_{jn} S_{kn}^*}{S_{0n}}$$

In this formula the identity representation 0 plays a special rôle. If the conformal field theory is unitary there is a natural choice for this representation, namely the one with smallest conformal weight. However in work on fixed points of simple currents [11] we were confronted with S-matrices that had all the right properties to be a modular group representation, but did not correspond to a unitary CFT. I will not attempt to explain fixed point resolution here, but the main point is that one needs a set of matrices S_{fix} ad T_{fix} that form a representation of the modular group. In most cases these fixed point resolution matrices are themselves "S-matrices" of WZW-models, but there are exceptional cases where an unknown matrix had to be used. These matrices are labelled by two positive integers n and m, and could be identified for $n = 1$ and $m = 1$ with the matrices S of non-unitary minimal Virasoro theories. The matrix for $n = m = 2$ was constructed numerically* in [11], and could not be identified with any known CFT.

To identify a conformal field theory that might correspond to this matrix we tried to determine which of the six fields was the identity. These "CFT"'s, which still have not been identified, were tentatively called $\mathcal{B}_{m,n}$. Our hope was to use the fusion rule formula to determine which field to use. A priori one would expect that using the wrong field "0" would lead to non-integer, fractional or infinite fusion rule coefficients.

Surprisingly, in the example we considered any choice seemed to be equally good. In all cases the fusion rule coefficients turned out to be integer, though in no cases

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$\star$ meanwhile it is known how to construct these matrices for arbitrary n and m [12].

were they positive. However, it was always possible to find a set of signs $\epsilon(i)$ and a new matrix $S'_{ij} = \epsilon(i)\epsilon(j)S_{ij}$ that made all the coefficients positive. There was a second surprise: the fusion rules turned out to be identical up to some permutation of all the fields.

Inspired by this observation we investigated a few WZW models in a similar way, and we found that some of the fields could play the rôle of the identity in the above sense. This was not pursued further at the time until recently, after the construction of the matrices S for $\mathcal{B}_{m,n}$ for arbitrary n and m (as far as we can tell now these have indeed such a symmetry under a certain signed cyclic permutation of all fields). Meanwhile the symmetry was discovered independently by Eholzer [13].

It is natural to look now for some underlying symmetry of S, and it turns out that indeed such a symmetry exists. Consider a matrix Π of the form

$$\Pi_{ij} = \epsilon(i)\delta_{i,\pi(j)} = \epsilon(\pi^{-1}(j))\delta_{\pi^{-1}i,j} ,$$

where π is a permutation and $\epsilon(i)$ a set of signs. Suppose the matrix S satisfies

$$S = \Pi S \Pi , \tag{1.2}$$

which implies that some elements of S are pairwise equal, up to signs. If one inserts this into the Verlinde formula, one finds

$$N_{ij}^k = \epsilon(i)\epsilon(j)\epsilon(k)\epsilon(0) \; N_{\pi(i)\pi(j)}^{\pi(k)}[\pi(0)] , \tag{1.3}$$

where the argument between square brackets indicates the choice of vacuum. This is exactly the behavior observed above. The signs can indeed be removed by redefining S in the way discussed above ($\epsilon(0)$ can be set equal to 1 by an overall sign choice).

One can easily imagine other symmetries of S that would lead to (1.3), for example one could replace one of the two matrices Π in (1.2) by any other signed permutation Π', but empirically (1.2) is the symmetry responsible for the observed relation (1.3). This relation among the element of S can also be written as

$$\Pi^T S = S\Pi , \quad \text{and} \quad \Pi S = S\Pi^T \tag{1.4}$$

because Π is an orthogonal matrix. This implies that the matrix $\Pi + \Pi^T$ commutes with S. This is a useful step towards finding modular invariant partition functions,

although unfortunately commutation with T and positivity are not automatically guaranteed.

In general S may have many symmetries of the type (1.2), but a very large and interesting subclass is obtained from Galois symmetry. In principle Galois symmetry of conformal field theory might have been discovered in this way, but actually it had already been found several years earlier by de Boer and Goeree [14]. The basic observation starts with writing the fusion rules in the following way

$$\frac{S_{i\ell}}{S_{0\ell}} \frac{S_{j\ell}}{S_{0\ell}} = \sum_k \mathcal{N}_{ij}^{\ k} \frac{S_{k\ell}}{S_{0\ell}} \tag{1.5}$$

This means that the *generalized quantum dimensions* $\frac{S_{j\ell}}{S_{0\ell}}$ furnish one-dimensional representations of the fusion algebra. Furthermore they exhaust the set of one-dimensional representations. Now suppose we have a map σ that takes complex numbers to complex numbers, that leaves integers fixed, and that respects sums and products, *i.e.* $\sigma(ab) = \sigma(a)\sigma(b)$ and $\sigma(a + b) = \sigma(a) + \sigma(b)$. Applying such a map to both sides of (1.5) we find then that $\sigma(\frac{S_{j\ell}}{S_{0\ell}})$ is a one-dimensional representation of the fusion algebra. Hence it must be one of the representations we already know, *i.e.* σ induces a permutation $\dot\sigma$ of the labels ℓ:

$$\sigma\left(\frac{S_{j\ell}}{S_{0\ell}}\right) = \frac{S_{j\dot\sigma(\ell)}}{S_{0\dot\sigma(\ell)}} \tag{1.6}$$

An example of a map with such properties is complex conjugation, but there is a more general class one can consider. These are the Galois transformations. Complex conjugation means interchanging the roots of the polynomial $x^2 + 1 = 0$. This polynomial has rational coefficients, but does not have roots within the rational numbers. One has to extend the field $\mathcal{Q}$ to $\mathcal{Q} + i\mathcal{Q}$ to solve the polynomial equation. This new field has a symmetry $a + ib \to a - ib$. This $\mathbf{Z}_2$ symmetry is called the Galois group of the field extension. This idea can be generalized to any polynomial with rational coefficients. For any such polynomial there is a minimal extension L of $\mathcal{Q}$ in which the polynomial has roots. For such an extension there exists a discrete group $\mathcal{G}(L/\mathcal{Q})$ of automorphisms of L that fix Q, called the Galois group. The elements of this group leave rational numbers invariant and act as permutations (possible trivial

ones) on the roots of the polynomial. Furthermore they respect the all defining properties of the field, in particular addition and multiplication. These symmetries can then be applied to obtain relations of the form (1.6).

In order to use Galois transformations it is important to know that the generalized quantum dimensions are indeed roots of some polynomial with rational coefficients. This is true because from (1.5) one sees that they are eigenvalues of the integer matrices $(N^i)_{jk} = N_{ij}^k$. Hence they are roots of the characteristic polynomials, and because the matrices are integer these polynomials do indeed have rational coefficients.

The next step was made by Coste and Gannon [15]. They observed that the matrix elements of S themselves belonged to a slightly larger extension of the rational numbers, containing L. Furthermore they showed that one can define a transformation similar to (1.6) for the matrix elements of S:

$$\sigma S_{i,\ell} = \epsilon_\sigma(\ell) S_{i,\dot\sigma(\ell)}$$

Since S is symmetric it is now easy to see that the Galois group must be abelian[$\star$] (two distinct elements can each act on a different index of S), and furthermore one sees immediately that

$$S_{ij} = \sigma\sigma^{-1}S_{ij} = \epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(j)S_{\dot\sigma(i)\dot\sigma^{-1}(j)} \ .$$

This is precisely (1.2).

The fact that the Galois group is abelian has an important consequence due to a theorem of Kronecker and Weber: it implies that the matrix elements of S are elements of a cyclotomic field. This is an extension of the rational numbers by the powers of a certain root of unity. Suppose the field is generated by $t = e^{\frac{2\pi i}{N}}$. Then Galois transformations correspond to sending $t \to t^l$ for some scale factor l. For this to be an automorphism of the field one has to require that l and N are relatively prime.

[$\star$] This had already been proved for $\mathcal{G}(L/\mathcal{Q})$ by de Boer and Goeree, but the proof is slightly more complicated in that case.

Up to now everything was valid for arbitrary conformal field theories. Let us now be more specific and discuss WZW-models. Then a formula exists for the matrix S :

$$S_{\lambda\mu} = \mathcal{N} \sum_w \epsilon(w) e^{2\pi i \frac{w(\lambda+\rho)\cdot(\mu+\rho)}{k+g}} \quad , \qquad (1.7)$$

where g is the dual Coxeter number and ρ the Weyl vector, whose Dynkin labels are $(1, 1, 1, 1, \ldots, 1, 1)$; $\mathcal{N}$ is a normalization. The sum is over the "horizontal" Weyl group (the Weyl group of the Lie algebra.) If $\mathcal{N}$ were rational the order N of the cyclotomic field would be determined completely by the denominator of the exponent. In general $\mathcal{N}$ is not rational, but this only leads to a further extension of the cyclotomic field that is not relevant for our purposes. The labels λ and μ in (1.7) are the highest weights of two representations. The order N of the cyclotomic field (as defined above) is thus $M(k + g)$, where M is the common denominator of the inner products among weights. Galois transformations correspond thus to those rescalings l of the exponent so that l and $M(k + g)$ are relatively prime.

A familiar result for ordinary simple Lie algebras is that any weight can be rotated into any preferred Weyl chamber by some suitable Weyl rotation. This is not different for Kac-Moody (affine) algebras. Only in this case there is an extra restriction on the Weyl chambers, which depends on k. Due to this restriction the affine Weyl chambers are finite in size. This is illustrated in fig.1 (next page), where the big triangle shows one affine Weyl chamber for A_2 level 5. The Kac-Moody highest weights at level 5 are precisely the weights in this triangle, including the boundary. It is again true that any weight can be transformed back into the affine Weyl chamber, but now one has to use the affine Weyl group, which contains in addition to the normal Weyl group also reflections with respect to the dashed line of the correct level (these lines are indicated in the figure).

If we were to ignore the shift by ρ in (1.7) we could transform the scaled weight $l\mu$ back to the affine Weyl chamber by means of affine Weyl transformations at level k. It is easy to see that horizontal Weyl transformations can be removed by shifting the sum over the horizontal Weyl group. This may change the sign of the overall factor $\epsilon(w)$ by a sign that depends on μ. However, the additional transformations in the affine Weyl group change the exponent unless they are generated by reflections with respect to the dashed line of level $k + g$.

The shift by ρ implies that we are not scaling μ but $\mu + \rho$. In the figure this is illustrated for $k = 2$: the shifted affine Weyl chamber for $k = 2$ is the dark triangle, and it is shown embedded in the unshifted $k = 5$ Weyl chamber (large triangle).
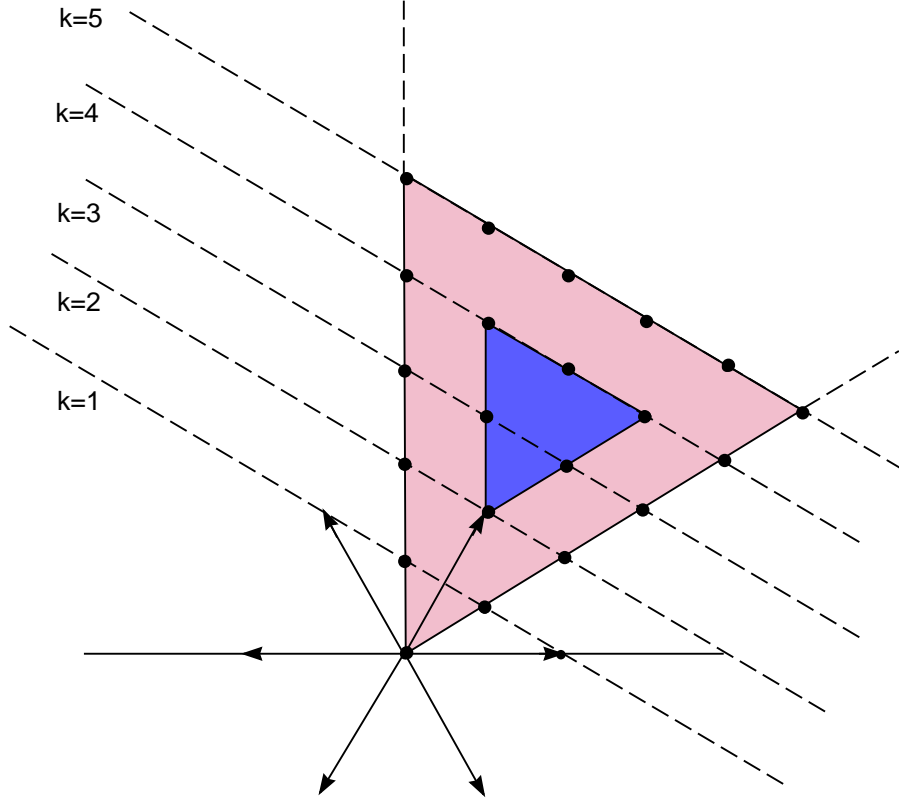
To map the scaled, shifted weight back to the dark triangle we can only use affine Weyl reflections at level $k + g$, if we want S to transform in a reasonable way. These transformations can be used to map all weights into the large triangle, but it would seem that nothing guarantees that we end up in the dark triangle. However, it is not hard to see that if the scale factor l is prime relative to $M(k + g)$ (which it must be due to the Galois condition) we do indeed end up in the dark triangle. In this way any highest weight μ is mapped to another highest weight, possibly accompanied by a sign change of S. In this way one can compute the matrices Π defined above. They can then be to construct elements in the integer commutant of S.

In [2] we have investigated the use of this knowledge for finding WZW modular invariants, and – to our surprise – we actually found some new invariants!

If l does not satisfy the Galois condition the map Π can not be expected to be

an automorphism. Indeed, one finds that it is not invertible, and in many cases some or all fields are transformed outside of the dark triangle. Those fields end up on the border of the large triangle. We can still represent the map by a matrix Π just as before, simply by ignoring the fields that are mapped to the border. Now the matrix Π will however be non-invertible. We call such scalings *quasi-Galois* scalings [3].

To illustrate the difference between Galois and quasi-Galois scalings, let us consider $G_{2,4}$. There are 9 primary fields, labelled as follows.

$$0 = (0,0); \quad 1 = (0,1); \quad 2 = (0,2);$$
$$3 = (0,3); \quad 4 = (0,4); \quad 5 = (1,0);$$
$$6 = (1,1); \quad 7 = (1,2); \quad 8 = (2,0).$$

Scaling by a factor 11, which is prime with respect to $N = 3(k+g) = 24$, we get a set of signs and permutations that can be represented by the following diagram. Here solid lines indicate a positive sign and dashed lines a negative one.
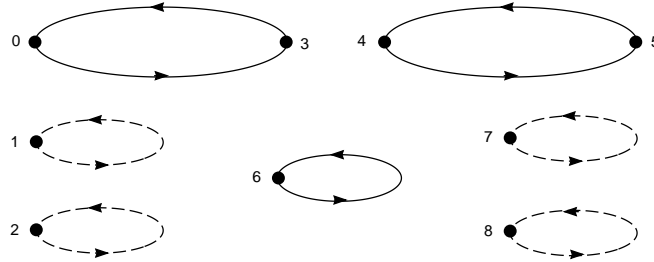
The matrix Π is obtained by setting to Π_{ij} to 1(-1) whenever there is a solid (dashed) arrow from i to j, and to zero if there is no line. This matrix is actually symmetric, and hence Π itself is an integer matrix commuting with S. It turns out that $\Pi + \mathbf{1}$ is a positive modular invariant, corresponding to the conformal embedding $G_{2,4} \subset SO(14)$.

The next picture illustrates a quasi-Galois scaling with $l = 2$, again for $G_{2,4}$. The grey blob represents the border of the positive Weyl chamber at level $k + g$. All other conventions are as before. It is quite clear that this picture represents a non-invertible map.
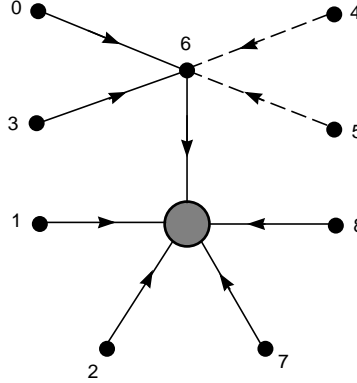
If we restrict ourselves to the five fields 0,3,4,5 and 6, the matrix Π has the following form

$$
\Pi = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
$$

and vanishes outside this subspace.

Remarkably, even though Π is not invertible, and though (1.2) does not hold, it turns out that (1.4) *does* hold. This is non-trivial and will be proved elsewhere [3]. An immediate consequence is that $\Delta = \Pi + \Pi^T$ commutes with S. Actually, it is not this matrix that is of most interest, but rather Δ^2, which of course also commutes with S. On the same subspace as before, it has the form

$$
\begin{pmatrix}
1 & 1 & -1 & -1 & 0 \\
1 & 1 & -1 & -1 & 0 \\
-1 & -1 & 1 & 1 & 0 \\
-1 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix} ,
$$

This is not a good modular invariant, but it turns out that the $G_{2,4}$ invariant described above can be obtained from it via a GSO-like projection. That invariant is explicitly:

$$M_{G_{2,4}} = |(0,0) + (0,3)|^2 + |(0,4+1,0)|^2 + 2|(1,1)|^2$$

The same quasi-Galois construction turns out to work for all conformal embeddings of a Kac-Moody algebra at level g into $SO(D)$, where D is the dimension of the adjoint representation (on the other hand, the Galois construction of the $G_{2,4}$ invariant is special for that algebra). In general they do not come out directly, but one gets a non-positive S, T^2 invariant from which the form of the desired invariant can be conjectured by applying a GSO-like projection. This is a useful result, since it is quite hard to construct those invariants by any other method.[*] For example, $E_{8,30}$ has 20956 primary fields, so that computing the matrix S is a horrendous task. Nevertheless we have been able to give a very plausible conjecture for the form of the invariant corresponding to the conformal embedding in $SO(248)$ using a quasi-Galois scaling with $l = 2$. We will discuss the details in [3], and we present in the appendix the results for the other exceptional algebras.

(Quasi)-Galois symmetry is yet another piece in the huge puzzle of classifying and describing (rational) conformal field theories. Most problems in this field remain unsolved. It is likely that the full potential of (quasi)-Galois symmetries has not been exploited yet. I hope that they can contribute to a more satisfactory state of affairs.

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[*] After [3] was first posted on "hep-th" we learned that Kac and Wakimoto [15] have presented – without any proof – formulas from which the decomposition of the identity and vector character can be derived. These formulas turn out to be equivalent to our results. No examples were given, however, and working them out with the formula in the form given in [15] would still be a laborious task.

APPENDIX

$M_{F_{4,9}} =$

$\quad + |(0,0,0,0) + (0,0,1,6) + (0,0,2,1) + (0,1,0,0)$

$\quad + (0,1,1,2) + (0,3,0,0) + (1,0,0,5) + (1,1,0,4)|^2$

$\quad + |(0,0,0,7,2) + (0,0,2,0,5) + (0,0,3,0,3) + (0,1,0,3,3)$

$\quad + (0,1,0,6,0) + (0,2,0,2,1) + (1,0,0,0,7) + (1,0,1,4,1)|^2$

$\quad + 2|2(1,1,1,1)|^2$

$M_{E_{6,12}} =$

$\quad + |(0,0,0,0,0,0) + (12,0,0,0,0,0) + (0,0,0,0,12,0) + (0,0,1,0,0,0)$

$\quad + (9,0,1,0,0,0) + (0,0,1,0,9,0) + (0,0,2,0,3,0) + (3,0,2,0,0,0)$

$\quad + (3,0,2,0,3,0) + (0,1,0,0,5,2) + (1,2,0,1,0,0) + (5,0,0,2,1,1)$

$\quad + (0,1,0,2,1,0) + (5,0,0,1,0,2) + (1,2,0,0,5,1) + (0,2,0,0,1,0)$

$\quad + (7,0,0,2,0,0) + (1,0,0,0,7,2) + (0,2,0,0,7,0) + (1,0,0,2,0,0)$

$\quad + (7,0,0,0,1,2) + (1,0,3,0,1,0) + (1,1,1,0,3,1) + (1,1,1,1,1,0)$

$\quad + (3,0,1,1,1,1) + (2,0,0,1,3,1) + (3,1,0,0,2,1) + (3,1,0,1,3,0)$

$\quad + (2,0,1,0,2,0) + (5,0,1,0,2,0) + (2,0,1,0,5,0) + (4,0,0,0,4,0)|^2$

$\quad + |(0,0,0,0,0,1) + (10,1,0,0,0,0) + (0,0,0,1,10,0) + (0,0,0,0,6,3)$

$\quad + (0,3,0,0,0,0) + (6,0,0,3,0,0) + (0,0,0,3,0,0) + (6,0,0,0,0,3)$

$\quad + (0,3,0,0,6,0) + (0,0,4,0,0,0) + (0,1,0,0,8,1) + (0,1,0,1,0,0)$

$\quad + (8,0,0,1,0,1) + (0,1,2,0,2,0) + (2,0,2,1,0,0) + (2,0,2,0,2,1)$

$\quad + (0,2,0,0,4,2) + (0,2,0,2,0,0) + (4,0,0,2,0,2) + (1,0,1,0,4,1)$

$\quad + (2,1,1,0,1,0) + (4,0,1,1,2,0) + (1,0,1,1,2,0) + (4,0,1,0,1,1)$

$\quad + (2,1,1,0,4,0) + (1,1,0,0,6,1) + (1,1,0,1,1,0) + (6,0,0,1,1,1)$

$\quad + (2,1,0,1,2,1) + (3,0,0,0,3,1) + (4,1,0,0,3,0) + (3,0,0,1,4,0)|^2$

$\quad + 2|4(1,1,1,1,1,1)|^2$

$$M_{E_{7,18}} =$$

$+ |(0,0,0,0,0,0,0) + (0,0,0,0,0,10,4) + (0,0,0,0,0,1,3) + (1,0,0,0,0,12,2)$
$+ (0,0,0,0,1,16,0) + (0,0,0,0,4,0,2) + (4,0,0,0,0,7,1) + (0,0,0,0,5,0,0)$
$+ (0,0,0,1,0,0,1) + (0,0,0,1,0,0,3) + (0,1,0,0,0,10,2) + (0,1,0,0,0,0,0)$
$+ (0,0,0,1,1,0,1) + (0,2,0,0,0,12,0) + (0,0,0,3,0,4,1) + (0,3,0,0,0,4,0)$
$+ (0,0,0,6,0,0,0) + (0,0,1,0,0,14,0) + (1,0,1,0,0,8,2) + (0,0,1,0,1,1,1)$
$+ (2,0,1,0,0,10,0) + (3,0,1,0,0,5,1) + (0,0,1,0,3,2,0) + (0,3,1,0,0,2,0)$
$+ (0,0,2,0,0,1,1) + (0,0,2,0,0,8,0) + (0,0,2,0,3,0,0) + (0,0,3,0,0,6,0)$
$+ (0,1,0,0,2,0,2) + (2,0,0,1,0,8,1) + (0,1,0,0,3,0,0) + (0,1,0,1,2,2,1)$
$+ (2,1,0,1,0,5,0) + (0,2,0,1,0,2,1) + (0,1,0,2,0,6,0) + (0,1,0,4,0,2,0)$
$+ (1,0,1,1,0,6,1) + (0,1,1,0,1,2,0) + (2,1,1,1,0,3,0) + (0,1,1,2,0,4,0)$
$+ (0,1,2,0,1,0,0) + (2,0,0,2,0,3,1) + (0,2,0,0,2,4,0) + (0,2,0,2,2,0,0)$
$+ (0,3,0,2,0,0,0) + (0,2,1,0,2,2,0) + (2,1,0,3,0,1,0) + (0,4,0,0,3,0,0)$
$+ (0,5,0,0,1,0,0) + (3,0,0,0,1,6,2) + (1,0,0,0,3,1,1) + (4,0,0,0,1,8,0)$
$+ (1,2,0,0,1,3,1) + (1,0,0,2,1,4,0) + (1,0,1,0,2,1,1) + (2,0,1,0,1,6,0)$
$+ (1,0,1,2,1,2,0) + (2,0,2,0,1,4,0) + (1,1,0,1,1,4,1) + (1,1,0,1,1,3,0)$
$+ (1,3,0,1,1,1,0) + (1,1,1,1,1,1,0) + (1,2,0,2,1,2,0) + (2,0,0,4,1,0,0)|^2$

$+ |(0,0,0,0,0,18,0) + (0,0,0,0,0,0,4) + (0,0,0,0,0,11,3) + (0,0,0,0,1,0,2)$
$+ (1,0,0,0,0,0,0) + (4,0,0,0,0,6,2) + (0,0,0,0,4,1,1) + (5,0,0,0,0,8,0)$
$+ (0,1,0,0,0,13,1) + (0,1,0,0,0,9,3) + (0,0,0,1,0,1,2) + (0,0,0,1,0,15,0)$
$+ (1,1,0,0,0,11,1) + (0,0,0,2,0,0,0) + (0,3,0,0,0,3,1) + (0,0,0,3,0,5,0)$
$+ (0,6,0,0,0,0,0) + (0,0,1,0,0,0,0) + (0,0,1,0,1,0,2) + (1,0,1,0,0,9,1)$
$+ (0,0,1,0,2,0,0) + (0,0,1,0,3,1,1) + (3,0,1,0,0,6,0) + (0,0,1,3,0,3,0)$
$+ (0,0,2,0,0,7,1) + (0,0,2,0,0,2,0) + (3,0,2,0,0,4,0) + (0,0,3,0,0,0,0)$
$+ (2,0,0,1,0,7,2) + (0,1,0,0,2,1,1) + (3,0,0,1,0,9,0) + (2,1,0,1,0,4,1)$
$+ (0,1,0,1,2,3,0) + (0,1,0,2,0,5,1) + (0,2,0,1,0,3,0) + (0,4,0,1,0,1,0)$
$+ (0,1,1,0,1,1,1) + (1,0,1,1,0,7,0) + (0,1,1,1,2,1,0) + (0,2,1,1,0,1,0)$
$+ (1,0,2,1,0,5,0) + (0,2,0,0,2,3,1) + (2,0,0,2,0,4,0) + (2,2,0,2,0,2,0)$
$+ (0,2,0,3,0,3,0) + (2,0,1,2,0,2,0) + (0,3,0,1,2,1,0) + (3,0,0,4,0,0,0)$
$+ (1,0,0,5,0,1,0) + (1,0,0,0,3,0,2) + (3,0,0,0,1,7,1) + (1,0,0,0,4,0,0)$
$+ (1,0,0,2,1,3,1) + (1,2,0,0,1,4,0) + (2,0,1,0,1,5,1) + (1,0,1,0,2,2,0)$
$+ (1,2,1,0,1,2,0) + (1,0,2,0,2,0,0) + (1,1,0,1,1,2,1) + (1,1,0,1,1,5,0)$
$+ (1,1,0,3,1,1,0) + (1,1,1,1,1,3,0) + (1,2,0,2,1,0,0) + (1,4,0,0,2,0,0)|^2$

$+ |8(1,1,1,1,1,1,1)|^2$

$$M_{E_{8,30}} =$$

$$
\begin{aligned}
&|(0,0,0,0,0,0,0,0) + (0,0,0,0,0,0,0,3) + (0,0,0,0,0,0,0,10) + (0,0,0,0,0,0,7,0) \\
&+ (0,0,0,0,0,1,0,1) + (0,0,0,0,0,1,1,1) + (0,0,0,0,1,0,1,1) + (0,0,0,0,1,0,5,0) \\
&+ (0,0,0,0,4,0,0,0) + (0,0,0,1,0,0,0,0) + (0,0,0,1,0,0,3,0) + (0,0,0,1,0,4,0,0) \\
&+ (0,0,0,1,1,0,1,0) + (0,0,0,2,0,0,4,0) + (0,0,0,3,0,0,0,0) + (0,0,0,3,0,0,0,4) \\
&+ (0,0,0,6,0,0,0,0) + (0,0,1,0,0,0,3,1) + (0,0,1,0,0,2,3,0) + (0,0,1,1,0,1,1,0) \\
&+ (0,0,2,0,0,3,0,1) + (0,0,2,0,1,0,3,0) + (0,0,2,0,3,0,0,0) + (0,0,2,1,0,1,0,0) \\
&+ (0,0,3,0,0,0,0,6) + (0,0,4,0,0,0,0,1) + (0,0,4,0,0,0,3,0) + (0,1,0,0,0,0,0,0) \\
&+ (0,1,0,0,0,0,5,0) + (0,1,0,0,1,0,3,0) + (0,1,0,0,2,2,0,0) + (0,1,0,1,0,2,2,0) \\
&+ (0,1,0,1,2,0,0,2) + (0,1,0,2,0,0,2,0) + (0,1,0,4,0,0,0,2) + (0,1,1,0,0,2,1,0) \\
&+ (0,1,1,2,0,0,0,4) + (0,1,2,0,0,1,2,1) + (0,1,2,0,1,0,1,0) + (0,1,2,0,1,2,0,0) \\
&+ (0,1,4,0,0,0,1,0) + (0,2,0,0,2,0,2,0) + (0,2,0,1,0,2,0,0) + (0,2,0,1,0,2,0,2) \\
&+ (0,2,0,2,2,0,0,0) + (0,2,1,0,2,0,0,2) + (0,2,2,0,0,1,0,1) + (0,2,2,0,1,0,2,0) \\
&+ (0,3,0,0,2,0,0,0) + (0,3,0,1,0,0,2,2) + (0,3,0,2,0,2,0,0) + (0,3,1,0,0,2,0,2) \\
&+ (0,3,2,0,1,0,0,0) + (0,4,0,0,3,0,0,0) + (0,4,0,1,0,0,0,2) + (0,4,0,2,0,0,2,0) \\
&+ (0,4,1,0,0,0,2,2) + (0,5,0,0,1,2,0,0) + (0,5,0,2,0,0,0,0) + (0,5,1,0,0,0,0,2) \\
&+ (0,6,0,0,1,0,2,0) + (0,7,0,0,1,0,0,0) + (0,8,0,0,0,0,3,0) + (0,9,0,0,0,0,1,0) \\
&+ (1,0,0,0,0,0,5,1) + (1,0,0,0,0,4,1,0) + (1,0,0,1,0,1,3,0) + (1,0,1,0,0,0,0,8) \\
&+ (1,0,1,0,1,2,1,0) + (1,0,1,0,2,1,0,1) + (1,0,1,1,0,0,0,6) + (1,0,1,1,0,1,2,0) \\
&+ (1,0,1,2,1,0,0,2) + (1,0,3,0,0,0,2,1) + (1,0,3,0,0,2,1,0) + (1,1,0,0,0,3,1,0) \\
&+ (1,1,0,1,1,0,0,4) + (1,1,1,0,1,1,1,0) + (1,1,1,0,1,1,1,1) + (1,1,1,1,1,1,0,1) \\
&+ (1,1,3,0,0,1,1,0) + (1,2,0,0,1,1,0,3) + (1,2,0,2,1,0,0,2) + (1,2,1,0,1,0,1,1) \\
&+ (1,2,1,1,0,1,1,1) + (1,3,0,0,0,1,1,3) + (1,3,0,1,1,1,0,1) + (1,3,1,1,0,0,1,1) \\
&+ (1,4,0,0,0,0,1,3) + (1,4,0,1,0,1,1,1) + (1,5,0,1,0,0,1,1) + (1,6,0,0,0,2,1,0) \\
&+ (1,7,0,0,0,1,1,0) + (2,0,0,0,3,0,1,0) + (2,0,0,1,0,3,0,0) + (2,0,0,2,0,1,0,3) \\
&+ (2,0,0,4,1,0,0,0) + (2,0,2,0,0,2,0,1) + (2,0,2,0,1,0,0,4) + (2,0,2,0,2,0,1,0) \\
&+ (2,1,0,0,2,1,0,0) + (2,1,0,1,1,0,1,2) + (2,1,0,3,0,1,0,1) + (2,1,1,1,0,1,0,3) \\
&+ (2,1,2,0,1,1,0,0) + (2,2,0,1,0,1,0,2) + (2,2,0,2,1,0,1,0) + (2,2,1,0,1,0,1,2) \\
&+ (2,3,0,2,0,1,0,0) + (2,3,1,0,0,1,0,2) + (2,4,0,0,2,0,1,0) + (2,5,0,0,1,1,0,0) \\
&+ (3,0,0,0,1,0,0,6) + (3,0,1,0,0,1,0,5) + (3,0,1,0,2,0,0,1) + (3,0,1,2,0,0,1,2) \\
&+ (3,1,0,1,0,0,1,4) + (3,1,1,1,1,0,0,1) + (3,2,0,0,1,0,0,3) + (3,2,0,2,0,0,1,2) \\
&+ (3,3,0,1,1,0,0,1) + (4,0,0,2,0,0,0,3) + (4,0,0,4,0,0,1,0) + (4,0,2,0,0,0,1,4) \\
&+ (4,1,0,3,0,0,0,1) + (4,1,1,1,0,0,0,3) + (5,0,0,0,0,0,1,6) + (5,0,1,0,0,0,0,5)|^2 \\
&+ \dots
\end{aligned}
$$

$$\ldots + |(0,0,0,0,0,0,1,2) + (0,0,0,0,0,0,6,1) + (0,0,0,0,0,1,0,2) + (0,0,0,0,0,2,0,0)$$
$$+ (0,0,0,0,0,5,0,0) + (0,0,0,0,1,0,0,0) + (0,0,0,0,1,0,2,0) + (0,0,0,0,2,0,0,0)$$
$$+ (0,0,0,1,0,0,2,1) + (0,0,0,1,0,1,4,0) + (0,0,0,2,0,1,0,0) + (0,0,1,0,0,0,0,0)$$
$$+ (0,0,1,0,0,0,4,0) + (0,0,1,0,1,0,2,0) + (0,0,1,0,1,3,0,0) + (0,0,1,0,3,0,0,1)$$
$$+ (0,0,1,1,0,1,3,0) + (0,0,1,2,0,0,1,0) + (0,0,1,3,0,0,0,3) + (0,0,2,0,0,0,0,7)$$
$$+ (0,0,2,0,0,2,0,0) + (0,0,3,0,0,0,3,1) + (0,0,3,0,0,3,0,0) + (0,0,3,0,1,0,0,0)$$
$$+ (0,0,5,0,0,0,0,0) + (0,1,0,0,0,0,0,9) + (0,1,0,0,0,0,4,1) + (0,1,0,0,0,3,2,0)$$
$$+ (0,1,0,1,0,1,2,0) + (0,1,0,2,0,0,0,5) + (0,1,1,0,1,1,2,0) + (0,1,1,0,1,2,0,1)$$
$$+ (0,1,1,1,0,1,1,0) + (0,1,1,1,2,0,0,1) + (0,1,3,0,0,0,1,1) + (0,1,3,0,0,1,2,0)$$
$$+ (0,2,0,0,0,3,0,0) + (0,2,0,0,2,0,0,3) + (0,2,0,3,0,0,0,3) + (0,2,1,0,1,0,2,1)$$
$$+ (0,2,1,0,1,1,0,0) + (0,2,1,1,0,2,0,1) + (0,2,3,0,0,1,0,0) + (0,3,0,0,0,2,0,3)$$
$$+ (0,3,0,1,2,0,0,1) + (0,3,1,0,1,0,0,1) + (0,3,1,1,0,0,2,1) + (0,4,0,0,0,0,2,3)$$
$$+ (0,4,0,1,0,2,0,1) + (0,4,1,1,0,0,0,1) + (0,5,0,0,0,0,0,3) + (0,5,0,1,0,0,2,1)$$
$$+ (0,6,0,0,0,3,0,0) + (0,6,0,1,0,0,0,1) + (0,7,0,0,0,1,2,0) + (0,8,0,0,0,1,0,0)$$
$$+ (0,10,0,0,0,0,0,0) + (1,0,0,0,0,0,0,0) + (1,0,0,0,0,0,6,0) + (1,0,0,0,1,0,4,0)$$
$$+ (1,0,0,0,3,1,0,0) + (1,0,0,1,0,3,1,0) + (1,0,0,2,0,0,3,0) + (1,0,0,2,1,0,0,3)$$
$$+ (1,0,0,5,0,0,0,1) + (1,0,1,0,0,2,2,0) + (1,0,2,0,0,2,1,1) + (1,0,2,0,1,0,2,0)$$
$$+ (1,0,2,0,2,1,0,0) + (1,0,2,1,0,0,0,5) + (1,0,4,0,0,0,2,0) + (1,1,0,0,2,1,1,0)$$
$$+ (1,1,0,1,0,2,1,0) + (1,1,0,1,1,1,0,2) + (1,1,0,3,1,0,0,1) + (1,1,1,1,1,0,0,3)$$
$$+ (1,1,2,0,0,1,1,1) + (1,1,2,0,1,1,1,0) + (1,2,0,0,2,0,1,0) + (1,2,0,1,0,1,1,2)$$
$$+ (1,2,0,2,1,1,0,0) + (1,2,1,0,1,1,0,2) + (1,2,2,0,1,0,1,0) + (1,3,0,1,0,0,1,2)$$
$$+ (1,3,0,2,0,1,1,0) + (1,3,1,0,0,1,1,2) + (1,4,0,0,2,1,0,0) + (1,4,0,2,0,0,1,0)$$
$$+ (1,4,1,0,0,0,1,2) + (1,5,0,0,1,1,1,0) + (1,6,0,0,1,0,1,0) + (1,8,0,0,0,0,2,0)$$
$$+ (2,0,0,0,0,4,0,0) + (2,0,0,1,0,0,0,7) + (2,0,1,0,1,0,0,5) + (2,0,1,0,1,2,0,0)$$
$$+ (2,0,1,0,2,0,1,1) + (2,0,1,2,0,1,0,2) + (2,0,3,0,0,2,0,0) + (2,1,0,1,0,1,0,4)$$
$$+ (2,1,1,0,1,1,0,1) + (2,1,1,1,1,0,1,1) + (2,2,0,0,1,0,1,3) + (2,2,0,2,0,1,0,2)$$
$$+ (2,2,1,1,0,1,0,1) + (2,3,0,0,0,1,0,3) + (2,3,0,1,1,0,1,1) + (2,4,0,1,0,1,0,1)$$
$$+ (2,6,0,0,0,2,0,0) + (3,0,0,0,3,0,0,0) + (3,0,0,2,0,0,1,3) + (3,0,0,4,0,1,0,0)$$
$$+ (3,0,2,0,0,1,0,4) + (3,0,2,0,2,0,0,0) + (3,1,0,1,1,0,0,2) + (3,1,0,3,0,0,1,1)$$
$$+ (3,1,1,1,0,0,1,3) + (3,2,0,2,1,0,0,0) + (3,2,1,0,1,0,0,2) + (3,4,0,0,2,0,0,0)$$
$$+ (4,0,0,0,0,1,0,6) + (4,0,1,0,0,0,1,5) + (4,0,1,2,0,0,0,2) + (4,1,0,1,0,0,0,4)$$
$$+ (4,2,0,2,0,0,0,2) + (5,0,0,4,0,0,0,0) + (5,0,2,0,0,0,0,4) + (6,0,0,0,0,0,0,6)|^2$$

$$+ 2|8(1,1,1,1,1,1,1,1)|^2$$

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